The Augmented Factorization Bound for Maximum-Entropy Sampling

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Maximum Entropy Sampling Problem (MESP)

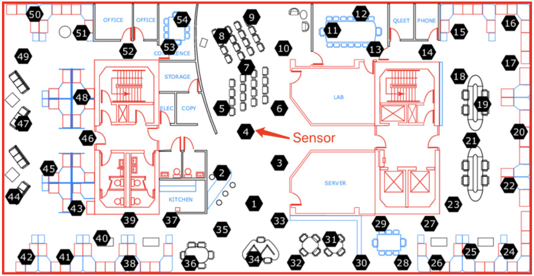

Figure: A 54-node network to measure the temperature (Bodik et al., 2004))

Sensor placement problem

- $[n] = \{1, 2, \cdots, n\}$: n available locations
- $x \in \mathbb{R}^n$: n random variables, e.g., temperature at n locations
- Goal: Choose a subset $S \subseteq [n]$, with $|S| = s$, to place sensors so that observing x_S maximizes the "information" about x

Maximum Entropy Sampling Problem (MESP)

- Entropy measures information (Shannon, 1948)
- Suppose that x follows Gaussian distribution with covariance matrix C. Then, the entropy obtained from observing x_S is

$$h(x_S) = \frac{1}{2}\log\det(C_{S,S}) + \frac{1}{2}\left(1 + \log(2\pi)\right)s$$

where

- $C_{S,S}$: A principal submatrix of C indexed by S
- $\log\det$: The natural logarithm of the determinant function

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$$\text{(MESP)} \quad v^* := \max_S \left\{\log\det\left(C_{S,S}\right) : S \subseteq [n],\ |S| = s\right\}$$

- MESP is NP-hard (Ko et al., 1995)

Existing upper bounds of MESP

- Spectral bounds: Bound the determinant based on the properties of eigenvalues
  - (Ko et al., 1995, Anstreicher and Lee, 2004, Burer and Lee, 2007, ...)
- Convex relaxation bounds
  - NLP bound (Anstreicher et al., 1999)
  - BQP bound (Anstreicher, 2018)
  - Linx bound (Anstreicher, 2020)
  - Factorization bound (Nikolov 2015, Li and Xie, 2024, Chen et al. 2023).

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This paper improves upon the factorization bound

Factorization bound of MESP (Nikolov, 2015)

- Recall

$$(\text{MESP}) \quad v^* := \max_S \left\{\log\det\left(C_{S,S}\right) : S \subseteq [n],\ |S| = s\right\}$$

- Assume that the covariance matrix C is positive definite
- Factorize C: $C = A^\top A$

Lemma (Nikolov, 2015): Reformulation of MESP

$$(\text{MESP}) \quad v^* := \max_{z\in\{0,1\}^n} \left\{\log \overset{s}{\det}\left(A\text{Diag}(z)A^\top\right) : \sum_{i\in[n]} z_i = s\right\},$$

where function $\overset{s}{\det}$ is the product of s largest eigenvalues of a matrix.

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- However, this is not a convex program
- Its Lagrangian dual yields the factorization bound

The augmented factorization bound of MESP

- First, subtract a scaled identity matrix tI_n from C, where $0 \leq t \leq \lambda_{\min}(C)$
- Second, factorize the positive semidefinite matrix $C - tI_n$
- Finally, derive the Lagrangian dual of MESP

Theorem

As t increases, the Lagrangian dual bound becomes tighter.

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Theorem

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- The factorization bound: $t = 0$
- The augmented factorization bound: $t = \lambda_{\min}(C)$
- The improvement depends on the condition number of C

Three benchmark datasets: $n = 63, 90, 124$

- Fact: The factorization bound (Nikolov, 2015)
- Linx: The linx bound (Anstreicher 2020)
- Mix-LF: Combine Fact and Linx (Chen et al., 2023)
- Aug-Fact: Our augmented factorization bound
- Integrality gap := Upper bound − Optimal value

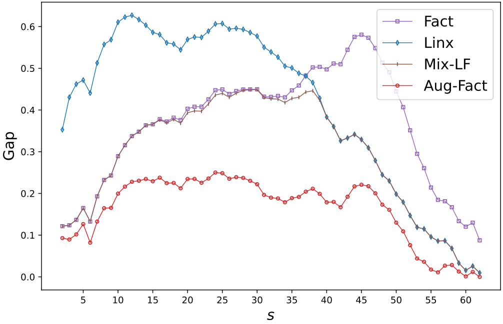

Figure: Integrality gap of $n = 63$, where the condition number of C is 48.42

Three benchmark datasets: $n = 63, 90, 124$

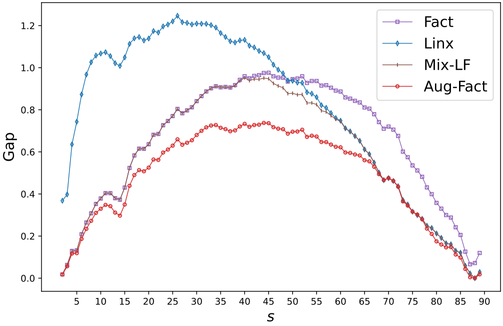

Figure: Integrality gap of $n = 90$, where the condition number of C is 200.45

Three benchmark datasets: $n = 63, 90, 124$

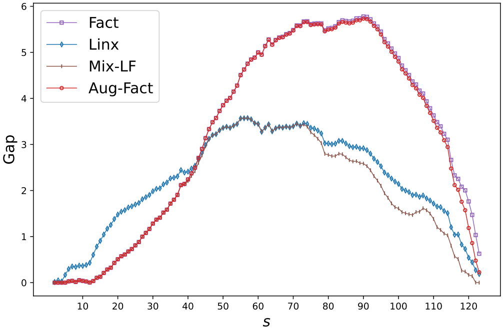

Figure: Integrality gap of $n = 124$, where the condition number of C is 78340.48

Thank you!

https://arxiv.org/pdf/2410.10078. Accepted at IPCO 2025